

# The AND-OR game: Equilibrium Characterization (working paper)

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**Abstract.** We consider a simple simultaneous first price auction for multiple items in a complete information setting. Our goal is to completely characterize the mixed equilibria in this setting, for a simple, yet highly interesting, **AND-OR** game, where one agent is single minded and the other is unit demand.

## 1 Introduction

Walrasian equilibrium is one of the most basic models in economic theory. Items are priced in such a way that for each item either the market clears (supply equals demand) or if there is an excess supply it is priced at zero. When there is a Walrasian equilibrium, it captures nicely the “right” pricing of items. Unfortunately, Walrasian equilibria are guaranteed to exist only for limited classes of agents’ valuations, namely gross-substitute valuations.

A different way of presenting the market is to auction the items simultaneously, and analyze the resulting equilibria. For simultaneous first price auction, the resulting pure Nash equilibria are in one-to-one correspondence to the Walrasian equilibria, with the same prices and allocations [3, 2]. Considering the market as a simultaneous first price auction allows us to consider it as a game, and study the resulting equilibria. Fortunately, there is always a mixed Nash equilibrium, with some tie breaking rule, and approximate Nash equilibrium with any tie breaking rule [3].

A typical example of a case where there are no Walrasian equilibria is when there are two agents, one is single minded while the other is unit demand. The **AND-OR** game is exactly this setting, with two items. The **AND** valuation is 1 if it gets both items and zero otherwise, while the **OR** has a value of  $v$  for any single item (or both) and zero otherwise. For  $v > 1/2$  there is no Walrasian equilibrium (or equivalently, pure Nash equilibrium) and this is the interesting and challenging case we focus on in this paper. A specific mixed Nash equilibrium for the **AND-OR** game was presented in [3] (we review it in Section 2).

In this work we completely characterize the resulting mixed Nash equilibria of the **AND-OR** game. We show that while the equilibrium is technically not

unique, it is almost unique. More precisely we show that all mixed Nash equilibria to the AND-OR game have the same marginal bid distributions for each item. The resulting prices and utilities of the AND and OR agents are the same in all mixed Nash equilibria. We complement our characterization with a study of the properties of these equilibria.

**Related Work:** There is a recent interest in algorithmic game theory to study simple simultaneous and sequential auctions to allocate multiple goods. Bhawalkar and Roughgarden [1] studied second price simultaneous auctions and their Price of Anarchy (PoA). They show that under the assumption of conservative bidding the PoA for sub-modular valuations is a constant and for sub-additive valuations it is logarithmic.

Hassidim et al. [3] studied a market which is based on first price auctions. They show that the pure equilibria correspond to Walarsian prices, and prove that a mixed Nash equilibrium always exists<sup>4</sup>. Similar to [1], they show for sub-modular valuations a constant PoA and for sub-additive valuations a logarithmic PoA.

Leme et al. [4] studied a sequential auctioning of the items, where the solution concept is a sub-game perfect equilibrium. They show that a pure sub-game perfect equilibrium always exists, and that for unit demand buyers the PoA is at most 2 while for submodular buyers it might be unbounded.

Szentes [5] studied a game with two identical bidders and two identical items in the full information model, where both bidders are either sub-additive or super-additive. In addition to simple pure Nash equilibria, he exhibits a family of symmetric mixed Nash equilibria. This work was extended in [7] to three items (where each of the two agent desires at least two of the items) and in [6] to multiple items and agents (where again, each agent desires a strict majority of the items). All the above works exhibit specific symmetric mixed equilibria, and none of them address the characterization of all mixed Nash equilibria.

## 2 The AND-OR Game: Model and preliminaries

We have two players an AND player and OR player. The AND player has a value of 1 if he gets both the items in  $M = \{1, 2\}$ , and the OR player has a value of  $v$  if she gets any item in  $M$ . Formally,  $v_{and}(M) = 1$  and for  $S \neq M$  we have  $v_{and}(S) = 0$ , also,  $v_{or}(T) = v$  for  $T \neq \emptyset$  and  $v_{or}(\emptyset) = 0$ . Both players have a quasi-linear utility with respect to money, i.e., getting subset  $S$  and paying price  $p$  has a utility of  $u(S) = v(S) - p$ , and are risk neutral. We assume a full information setting, namely both players know each other's valuation.

In the AND-OR game both players participate in two simultaneous first price auctions, one for each item. Namely, each player places bids  $(x_1, x_2)$  where  $x_i$  is the bid on item  $i$ . The highest bidder on each item wins it and pays its bid. We will denote by  $H$  the maximum allowed bid in the game, so  $0 \leq x, y \leq H$ . (Clearly it suffices to have  $H \leq \max(1, v)$ .) Completely specifying a first price

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<sup>4</sup> the issue is that the prices are continuous.

auction requires a *tie breaking rule*, which specifies the winner in case of identical bids. We make the assumption that the tie breaking rule only depends on the bids for the tied item (and not on the bids on the other item), and allow for a randomized tie breaking rule.

When  $v \leq 1/2$  there is a Walresian equilibrium for any price  $p \in [v, 1/2]$  per item. This implies a pure Nash Equilibrium in which both players bid  $p$  on each item, and the AND player wins both items (assuming the tie breaking rule favors AND). For this reason we are interested in the case when  $v > 1/2$ . It is easy to verify that in this case is no Walresian equilibrium. For completeness we show that there is no pure Nash equilibrium.

*Claim.* There is no pure Nash equilibrium in the AND-OR game.

*Proof.* Assume for contradiction there was a pure Nash equilibrium with some tie breaking rule. Let the AND bid  $(x, y)$ . Since the AND value is 1, if  $x + y > 1$  the AND has a negative utility for any best response of the OR. In the case that  $x + y \leq 1$ , the best response of the OR is to out bid the lower bid of the AND. Therefore, AND will have a non-negative utility only if it bids  $(0, 0)$ . In case that AND bids  $(0, 0)$ , the OR wins one item at a price of at most  $\epsilon$ . But then the AND can deviate and bid  $(2\epsilon, 2\epsilon)$  and have a positive utility. Contradiction.  $\square$

Next, we describe the mixed Nash equilibrium from [3].

- The AND player bids  $(y, y)$  where  $0 \leq y \leq 1/2$  according to cumulative distribution  $F_{and}^*(y) = (v - 1/2)/(v - y)$  (where  $F_{and}^*(y) = Pr[bid \leq y]$ ). In particular, There is an atom at 0:  $Pr[y = 0] = 1 - 1/(2v)$ .
- The OR player bids  $(x, 0)$  with probability  $1/2$  and  $(0, x)$  with probability  $1/2$ , where  $0 \leq x \leq 1/2$  is distributed according to cumulative distribution  $F_{or}^*(x) = x/(1 - x)$ .

Note that since the OR player does not have any mass points in his distribution, the equilibrium holds for any tie breaking rule. The proof that this is indeed an equilibrium is in [3]. The main goal of this paper is to characterize the mixed Nash equilibria of the AND-OR game, and to show that this is “essentially” the only mixed Nash equilibrium.

### 3 Characterization of the mixed Nash equilibria

The following theorem is our main result which characterizes the mixed Nash equilibria of the AND-OR game. The OR player has to play the mixed strategy  $F_{or}^*$ . The AND player can play various mixed strategies, but their marginal bid distribution on each item is identical to  $F_{and}^*$ , and the probability mass at  $(0, 0)$  is the same as of  $F_{and}^*$ . While there is more than a single equilibrium, they all have “essentially” the same outcomes, i.e., the same expected utilities, payments, and allocation probabilities.

**Theorem 1.** *A pair of strategies  $(F_{and}, F_{or})$  is an equilibrium of the AND-OR game if and only if*

1. OR's strategy is  $F_{or} = F_{or}^*$ .
2. AND's strategy has the same marginal distributions of the bids for each item as  $F_{and}^*$ :  $F_{and}(x, H) = F_{and}^*(x, H)$  and  $F_{and}(H, y) = F_{and}^*(H, y)$  for all  $x, y$ , and the same probability of  $(0, 0)$ :  $F_{and}(0, 0) = F_{and}^*(0, 0)$ .

Furthermore,  $F_{and}$  is weakly dominated as a strategy by  $F_{and}^*$  and the following quantities are the same as in the equilibrium  $(F_{and}^*, F_{or}^*)$ : (1) The allocation probabilities, i.e. the probabilities of each player winning each bundle. (2) The expected payments, and utilities of each player. Thus also the expected revenue and social welfare.

The proof of the above theorem is quite involved, and most of the paper is devoted to it. In the following we give a high level view of the proof. We start with preliminary set-up. In Section 3.1 we show that the support of the bids of each player for each item is (essentially) an interval, with both players having the same upper bound. In Section 3.2 we observe that OR must get non-zero utility. In Section 3.3 we show that without loss of generality we can increase the correlation between the bids of AND on the two items without changing its marginal distribution on each item. This allows us to consider maximally correlated distributions for AND. (We formally define in Section 3.3 what we mean by maximally correlated.) In Section 3.4 we prove that the bids of OR must be on the axis: i.e. either  $(0, x)$  or  $(x, 0)$ ; and then in Section 3.5 we show that AND must bid on the diagonal, i.e.,  $(x, x)$ . At this point we derive, in Section 3.6 the exact form of the equilibrium distributions. Section 3.7 puts everything back together showing how the main theorem is implied.

### 3.1 The Bids on Each Item form an Interval

The results in this section apply to any equilibrium in a market with two bidders and any number of items. We still state these results using the names AND and OR for the players but we only assume that  $(F_{and}, F_{or})$  are an equilibrium of some market game. We do not rely in this section on the specific form of the utilities of AND and OR, but only on the fact that our game is a simultaneous first price auction. This implies that Lemma 2 also holds with the roles of AND and OR reversed.

**Lemma 1.** *The highest bids of the two players on a particular item are equal. I.e.  $F_{and}(x, H) = 1$  if and only if  $F_{or}(x, H) = 1$  (and similarly for the other items.)*

*Proof.* Assume that  $F_{or}(x_0, H) = 1$  for some  $x_0$ . For any bid  $(x, y)$  of AND, with  $x > x_0$ , AND gets strictly lower utility than  $(x_0 + \epsilon, y)$  (for any  $\epsilon < x - x_0$ ) since it wins in exactly the same cases  $(x, y)$  won, and always pays strictly less. It follows that no bid with  $x > x_0$  is a best-response for AND, thus AND always bids at most  $x_0$  on the first item, i.e.,  $F_{and}(x_0, H) = 1$ . The other direction is analogous.  $\square$

Given Lemma 1 it is possible to define the “highest bid” on an item:  $h_1 = \min\{x \mid F_{and}(x, H) = 1\}$  and  $h_2 = \min\{y \mid$

$F_{and}(H, y) = 1\} = \min\{y \mid F_{or}(H, y) = 1\}$  (where the minimum is actually achieved due to the right-continuity of CDFs.) We will argue below that in fact  $h_1 = h_2$ .

**Lemma 2.** *Let  $0 < b < c$  such that OR never bids between  $b$  and  $c$  on an item, i.e.,  $F_{or}(b, H) = F_{or}(c, H)$  and such that both players sometimes bid at most  $b$ , i.e.,  $F_{or}(b, H) > 0$  and  $F_{and}(b, H) > 0$ . Then both players always bid at most  $b$ , i.e.  $F_{or}(b, H) = F_{and}(b, H) = 1$ .*

*Proof.* First, if OR never bids above  $c$ , i.e.,  $F_{or}(c, H) = 1$ , then also  $F_{or}(b, H) = F_{or}(c, H) = 1$  and by Lemma 1 also  $F_{and}(b, H) = 1$ .

Otherwise, for a contradiction, assume OR bids above  $c$ . Define  $d$  to be the infimum bid of OR above  $b$ :  $d = \inf\{x \mid F_{or}(x, H) > F_{or}(b, H)\}$ . This implies that for any  $\epsilon > 0$  the OR bids in  $[d, d + \epsilon)$  with positive probability. We show a sequence of properties that depends on  $d$ :

(I) AND does not bid in the range  $(b, d)$  on that item (i.e.,  $F_{and}(x, H) = F_{and}(b, H)$  for all  $b \leq x < d$ ). Assume for a contradiction that AND bids  $(x, y)$  for some  $b < x < d$ . Now consider a deviation of AND that bids  $(b + \epsilon, y)$ . Both  $(x, y)$  and  $(b + \epsilon, y)$  win in exactly in the same cases, however, with the bid  $(b + \epsilon, y)$  AND pays strictly less (for any  $\epsilon < x - b$ ). Since this occurs with positive probability, we have a contradiction that  $(x, y)$  is a best response of AND.

(II) Assume AND player does not have an atom at  $d$ . Consider a deviation of the OR player switching any bid for the first item which is in the range  $(d, d + \epsilon)$  (for a small enough  $\epsilon \geq 0$ ), with the bid  $(b + d)/2$ . The probability that AND bids in  $(d, d + \epsilon)$  goes to zero as  $\epsilon$  goes to zero, since AND does not have an atom at  $d$ . This upper bounds the loss of OR in the deviation. On the other hand, the payments decrease by at least  $(d - b)/2 > 0$  and this happens with positive probability (at least the probability that AND bids below  $b$ ). Therefore we reached a contradiction to the assumption that OR is best responding.

(III) Assume AND has an atom at  $d$  and the OR does not have an atom at  $d$ . Consider a deviation where the AND switches the bid of  $d$  for the first item by the bid  $(b + d)/2$ . Since the OR does not have an atom at  $d$ , the probability that AND wins does not change, and the payments go down by  $(d - b)/2$  with constant probability (at least the probability that OR bids below  $b$ .) Therefore we reached a contradiction to the assumption that AND is best responding.

(IV) Assume both AND and OR have an atom at  $d$ . Now, look at the tie breaking rule at  $d$ , it gives AND probability  $q_d$  of winning the tie at  $d$  and OR a probability of  $1 - q_d$  winning. At least one of the players does not always win. That player may want to increase its bid to  $d + \epsilon$  and always win the item – this will be strictly beneficial unless its expected utility from bidding  $d$  is exactly zero. But in that case reducing its bid to  $b + \epsilon$  will strictly increase its utility: winning whenever he previously did and paying less (again, winning the item with these bids has positive probability.)

We reached a contradiction to the assumption that OR bids above  $c$ , therefore  $F_{or}(b, H) = 1$ . By Lemma 1 we have that  $F_{and}(b, H) = 1$ .  $\square$

### 3.2 OR Gets Positive Utility

**Lemma 3.** *In any mixed Nash equilibrium  $(F_{and}, F_{or})$  the expected utility of the OR player is strictly positive.*

*Proof.* For contradiction assume that the expected utility of the OR player is zero, i.e.,  $u_{or}(F_{and}, F_{or}) = 0$ . Consider the following deviation of the OR player. Let  $\eta = (0.5 + v)/2 > 1/2$ . The OR player bids with probability  $1/2$  the bids  $(0, \eta)$  and with probability  $1/2$  the bids  $(\eta, 0)$ . Since we are at an equilibrium the expected utility of this deviation is 0. This could happen only if the AND player always bids in  $F_{and}$  above  $\eta$  for both items, i.e.,  $F_{and}(\eta, H) = F_{and}(H, \eta) = 0$ .

If the AND player always bids above  $\eta$  for both items then he has a non-negative utility only if he always loses both items. It follows that the OR player, using  $F_{or}$ , always wins both items with a cost of at least  $\eta$  for each. However, this implies that the OR player pays at least  $v + 0.5$ . Since the value of the OR player is  $v$  she has a negative utility, which is a contradiction.  $\square$

Let  $p_{or}(x, y)$  be the probability that the OR wins at least one item with the bid  $(x, y)$ , i.e.,  $p_{or}(x, y) = F_{and}(x, H) + F_{and}(H, y) - F_{and}(x, y)$ . The following simple corollary to Lemma 3 would be useful.

**Corollary 1.** *In any mixed Nash equilibrium  $(F_{and}, F_{or})$ , if  $F_{or}(x, y) > 0$  then  $p_{or}(x, y) > 0$ .*

Consider the highest bids  $h_1$  and  $h_2$ . Clearly if  $h_1 > h_2$  and  $h_2 < v$  then the OR can gain by deviating and bidding  $h_2 + \epsilon$  on the second item and 0 on the first item whenever it used to bid  $y > h_2$  on the first item. If  $h_1 > h_2 = v$  then OR cannot have positive utility contradicting Lemma 3. So we have the following corollary.

**Corollary 2.** *The highest bid of the players on the first item ( $h_1$ ) is equal to the highest bid of the players on the second item ( $h_2$ ).*

We let  $h = h_1 = h_2$ .

### 3.3 Identical Marginal Distributions and Correlation

It will be convenient to consider the projection of the joint bid distribution of a player on the individual coordinates, i.e., the single items.

**Definition 1.** *Two CDF's  $F$  and  $F'$  are called Identical Marginal Distributions if their marginals are identical. I.e.,  $F(x, H) = F'(x, H)$  for all  $x$  and  $F(H, y) = F'(H, y)$  for all  $y$ .*

The following proposition takes advantage of the fact that the decision of each auction depends only on the marginal distribution.

**Proposition 1.** *If  $F_{and}$  and  $F'_{and}$  are Identical Marginal Distributions then against any strategy  $F_{or}$ :*

- Each item is won by each player with the same probability in  $(F_{and}, F_{or})$  and  $(F'_{and}, F_{or})$ .
- The expected payments of each player are the same in  $(F_{and}, F_{or})$  and  $(F'_{and}, F_{or})$ .

Note that the above proposition states that the probabilities of winning any single item by any player are identical in  $(F_{and}, F_{or})$  and  $(F'_{and}, F_{or})$ , but the probability of winning both items might differ.

**Proposition 2.** *If  $F_{and}$  and  $F'_{and}$  are Identical Marginal Distributions then, for every  $F_{or}$ , the following conditions are equivalent:*

1.  $u_{and}(F_{and}, F_{or}) \leq u_{and}(F'_{and}, F_{or})$ .
2.  $u_{or}(F_{and}, F_{or}) \geq u_{or}(F'_{and}, F_{or})$ .
3.  $Pr_{(F_{and}, F_{or})}[\text{AND wins both items}] \leq Pr_{(F'_{and}, F_{or})}[\text{AND wins both items}]$ .
4.  $Pr_{(F_{and}, F_{or})}[\text{OR wins an item}] \geq Pr_{(F'_{and}, F_{or})}[\text{OR wins an item}]$
5.  $Pr_{(F_{and}, F_{or})}[\text{AND wins no items}] \leq Pr_{(F'_{and}, F_{or})}[\text{AND wins no items}]$ .

*Proof.* Since  $F_{and}$  and  $F'_{and}$  are Identical Marginal Distributions the expected payments are identical for both players. This implies that the only parameter that influences the utility is the probability of winning. (For the AND player, winning both items, for the OR player, winning one of the two items.) Thus (1) is equivalent to (3) and (2) is equivalent to (4). But notice that always exactly one of AND or OR wins, and therefore if the probability that AND wins increases, then the probability that OR wins decreases. It follows that (3) and (4) are equivalent. Finally, since  $F_{and}$  and  $F'_{and}$  are Identical Marginal Distributions the players win each item with the same probability. Let  $p_1$  be the probability that AND wins item 1 and  $p_2$  be the probability that AND wins item 2 (both with  $F_{and}$  and with  $F'_{and}$ ). Then we have  $Pr[\text{AND wins no item}] = 1 - p_1 - p_2 + Pr[\text{AND wins both items}]$  and thus (3) and (5) are equivalent.  $\square$

An immediate corollary is that if the utilities of one player are identical, then the utilities of the other player are also identical.

**Corollary 3.** *Assume that  $F_{and}$  and  $F'_{and}$  are Identical Marginal Distributions. Then  $u_{and}(F_{and}, F_{or}) = u_{and}(F'_{and}, F_{or})$  iff  $u_{or}(F_{and}, F_{or}) = u_{or}(F'_{and}, F_{or})$*

A very important building block in our proof is the notion of *maximally correlated*. Intuitively, if the support of a distribution is on a monotone increasing line, then this distribution is maximally correlated. Since we want to show that the AND player support is essentially the diagonal, this would be very useful to characterize its bid distribution.

**Definition 2.** *For a CDF  $F_{and}$  define  $\bar{F}_{and}(x, y) = \min(F_{and}(x, H), F_{and}(H, y))$ .  $F_{and}$  is called maximally correlated if  $F_{and} = \bar{F}_{and}$ .*

Note that  $\bar{F}_{and}(x, H) = \min(F_{and}(x, H), F_{and}(H, H)) = \min(F_{and}(x, H), 1) = F_{and}(x, H)$ , and therefore the following proposition holds.

**Proposition 3.** *Every CDF  $F_{and}$  is Identical Marginal Distribution to  $\bar{F}_{and}$ .*

The following proposition claims that  $\bar{F}_{and}$  stochastically dominates  $F_{and}$ .

**Proposition 4.** *For every  $(x, y)$ ,  $F_{and}(x, y) \leq \bar{F}_{and}(x, y)$ . I.e.  $F_{and}$  stochastically dominates  $\bar{F}_{and}$ .*

The following lemma shows that a maximally correlated strategy  $\bar{F}_{and}$  weakly dominates the original strategy  $F_{and}$ .

**Lemma 4.** *Every  $F_{and}$  is weakly dominated, as a strategy of AND, by  $\bar{F}_{and}$ .*

*Proof.* Fix a pure bid  $(x, y)$  of OR and let  $F_{or}$  bid  $(x, y)$  with probability 1. By Proposition 2 we have that  $u_{and}(F_{and}, F_{or}) \leq u_{and}(\bar{F}_{and}, F_{or})$  if and only if  $F_{and}(x, y) = Pr_{(F_{and}, F_{or})}[\text{AND wins both items}] \leq Pr_{(\bar{F}_{and}, F_{or})}[\text{AND wins both items}] = \bar{F}_{and}(x, y)$ . The latter holds, since by Proposition 4 we have  $F_{and}(x, y) \leq \bar{F}_{and}(x, y)$ .  $\square$

The following lemma shows that if we have an equilibrium  $(F_{and}, F_{or})$ , and we replace  $F_{and}$  by  $\bar{F}_{and}$ , then we still remain in an equilibrium. The main part of the proof is showing that the OR strategy remains a best response to  $\bar{F}_{and}$ .

**Lemma 5.** *If  $(F_{and}, F_{or})$  is a Nash equilibrium then  $(\bar{F}_{and}, F_{or})$  is a Nash equilibrium. Moreover it produces exactly the same distribution on allocations, payments, and utilities as does  $(F_{and}, F_{or})$ .*

*Proof.* By Lemma 4,  $\bar{F}_{and}$  dominates  $F_{and}$  so  $u_{and}(F_{and}, F_{or}) \leq u_{and}(\bar{F}_{and}, F_{or})$ . Since  $(F_{and}, F_{or})$  is an equilibrium, then  $u_{and}(F_{and}, F_{or}) = u_{and}(\bar{F}_{and}, F_{or})$  and by Corollary 3 we also have that  $u_{or}(F_{and}, F_{or}) = u_{or}(\bar{F}_{and}, F_{or})$ . Since  $F_{or}$  is best response to  $F_{and}$  we have that  $u_{or}(F_{and}, F_{or}) \geq u_{or}(F_{and}, F'_{or})$ , for any  $F'_{or}$ .

Since by Lemma 4 we also have that  $u_{and}(\bar{F}_{and}, F'_{or}) \geq u_{and}(F_{and}, F'_{or})$  then by Proposition 2,  $u_{or}(\bar{F}_{and}, F'_{or}) \leq u_{or}(F_{and}, F'_{or})$ . Thus,

$$u_{or}(\bar{F}_{and}, F'_{or}) \leq u_{or}(F_{and}, F'_{or}) \leq u_{or}(F_{and}, F_{or}) = u_{or}(\bar{F}_{and}, F_{or}),$$

which implies that  $F_{or}$  is a best response to  $\bar{F}_{and}$ . From Proposition 2 we get that the allocations, payments, and utilities are identical.  $\square$

We will continue the analysis assuming that  $F_{and} = \bar{F}_{and}$  is already maximally correlated and will derive the form of the equilibrium under this assumption. By Lemma 5 this characterization will then apply to other distributions that are Identical Marginal Distributions to  $F_{and}$ , and form an equilibrium with  $F_{or}$ . We will keep the notation  $\bar{F}_{and}$  to stress that it is maximally correlated.

### 3.4 OR Bids on the Axis

Our approach here is to first show that OR must always place a certainly-losing bid on one of the items; and then to show that AND bids arbitrarily low on each item, implying that certainly-losing bids of OR must be 0.

We start by defining the low bids for AND in a distribution  $F_{and}$ :



**Definition 3.**  $L_{and}^1 = \inf\{x \mid F_{and}(x, H) > 0\}$  and  $L_{and}^2 = \inf\{y \mid F_{and}(H, y) > 0\}$ .

Definition 5 specifies similar quantities for OR but in a *different* way.

**Definition 4.** Given a distribution of AND,  $F_{and}$ , we say that a bid  $x$  of OR for item  $i$  is low, if either  $x = 0$  or OR with bid  $x$  always loses item  $i$  to AND. We denote a low bid of OR by  $\ell_{or}^i$ .

By definition, a bid  $x > 0$  of OR is low if  $x < L_{and}^i$  or if  $x = L_{and}^i$  and either AND doesn't have an atom at  $L_{and}^i$  or AND always wins the tie for item  $i$  at  $L_{and}^i$ .

**Lemma 6.** Assume that  $(\bar{F}_{and}, F_{or})$  is an equilibrium, then the OR player bids a low bid on exactly one of the items. Moreover the probability of bidding low on each one of the items is positive.

*Proof.* Since  $\bar{F}_{and}$  is maximally correlated, we have that the probability that the OR wins at least one item using a bid  $(x, y)$  is  $\max\{\bar{F}_{and}(x, H), \bar{F}_{and}(H, y)\}$  (this holds since the probability of winning at least one item is  $p_{or}(x, y) = \bar{F}_{and}(x, H) + \bar{F}_{and}(H, y) - \bar{F}_{and}(x, y)$ , and since  $\bar{F}_{and}$  is maximum correlated  $\bar{F}_{and}(x, y) = \min\{\bar{F}_{and}(x, H), \bar{F}_{and}(H, y)\}$ ). Assume that  $\bar{F}_{and}(x, H) \geq \bar{F}_{and}(H, y)$ . Consider a deviation of the OR where she bids  $(x, 0)$  instead of  $(x, y)$ . The probability that OR wins at least one item is the same, i.e.,  $p_{or}(x, y) = p_{or}(x, 0)$ . The payment decreases by  $y$  times the probability that OR wins the second item. This is a strict decrease unless  $y$  is low. This establishes that in each bid  $(x, y)$  in the support of  $F_{or}$  at least one of  $x$  or  $y$  must be low.

We now want to show that  $F_{or}(\ell_{or}^1, \ell_{or}^2) = 0$  for any low bids  $\ell_{or}^1$  and  $\ell_{or}^2$ . Namely, the probability that the OR has both bids low is zero. Assume by contradiction that  $F_{or}(\ell_{or}^1, \ell_{or}^2) > 0$  for some low bids  $\ell_{or}^1$  and  $\ell_{or}^2$  and consider the following cases where OR bids  $(x, y)$  such that  $x \leq \ell_{or}^1$  and  $y \leq \ell_{or}^2$ :

(1) OR always loses both items. This implies that the OR has zero utility, contradicting Lemma 3.

(2) OR wins with positive probability. Since OR always loses with a low value which is not zero we may assume that either  $F_{or}(\ell_{or}^1, 0) > 0$  or  $F_{or}(0, \ell_{or}^2) > 0$ . We assume that  $F_{or}(\ell_{or}^1, 0) > 0$ , the case where  $F_{or}(0, \ell_{or}^2) > 0$  is symmetric. We also denote the probability that OR wins with a bid  $(x, 0)$  where  $x \leq \ell_{or}^1$  by  $p = p_{or}(\ell_{or}^1, 0) > 0$ . Consider now the a deviation for the AND player, in which it increases every bid on every item by  $\epsilon < p/3$ . This increases its probability of winning (and its expected welfare) by at least  $p$ , and increases its payments by  $2p/3$ , which gives a net utility gain of at least  $p/3$ .

Therefore, we have established that the OR always bids one low value and never bids both values low.

It remains to show that OR must place a low bid with positive probability on each item. By contradiction, assume that the OR always bids low only on one of the items, say item 1. We have two cases:

(1) The AND player always wins item 1. Let  $\bar{\ell}_{or}^1$  be the supremum of the (low) bids of OR on item 1. Then it is clear that in equilibrium the AND never bids

above  $\bar{\ell}_{or}^1 + \epsilon$  on item 1, for any  $\epsilon > 0$ . Now consider item 2. We have two cases depending on the relation between  $v$  and 1: (1a) If  $v < 1$  then (given that the AND always wins item 1 and no-matter in what price) the AND player will win item 2 and pay at most  $v + \epsilon$ . Since the AND player wins both items the OR player has zero utility, contradicting Lemma 3. (1b) If  $v > 1$  then the OR player always wins item 2. This implies that the AND player always loses, and has negative utility unless  $\bar{\ell}_{or}^1 = 0$ . However, in this case the OR player can deviate and bid  $(2\epsilon, 0)$  and always win item 1, contradicting the assumption that we have an equilibrium.

(2) The OR player wins item 1 with positive probability  $p > 0$ . This can happen only if  $\bar{\ell}_{or}^1 = 0$ . Let  $\delta$  denote the expected price of item 2. If  $\delta = 0$ , the AND player can strictly increase its utility by bidding  $(p/3, p/3)$ . If  $\delta > 0$ , then the auction has expected revenue at least  $\delta$ , and therefore the expected utility of the OR player is at most  $v - \eta$  for some  $\eta > 0$ . Since the OR player is always bidding 0 on item 1 the AND player will bid at most  $\epsilon = \eta/3$  on item 1 (this is true for any  $\epsilon > 0$ ). But then bidding  $(2\eta/3, 0)$  is a profitable deviation for the OR player.

We have established that the OR player has to bid a low bid with a positive probability on each item.  $\square$

Now let us define for OR  $L_{or}^i$ , which is different from the definition of AND.

**Definition 5.**  $L_{or}^1 = \inf\{x \mid F_{or}(x, y) > 0 \text{ and } y \text{ is low}\}$  and  $L_{or}^2 = \inf\{y \mid F_{or}(x, y) > 0 \text{ and } x \text{ is low}\}$ .

**Lemma 7.** Assume that  $(\bar{F}_{and}, F_{or})$  is an equilibrium and  $L_{and}^1 = L_{or}^1 = l$ . Then at most one of the players can have an atom at  $l$  in the marginal distribution of item 1.

*Proof.* Assume by contradiction that they both have atoms at  $l$ . By Lemma 6, the OR player never has both bids low, so  $L_{or}^1$  cannot be low. Since  $L_{or}^1$  is not low, the OR wins the tie with non-zero probability.

Now consider the following cases depending on  $L_{and}^1$ :

(1) If  $L_{and}^1 < 1$  then AND can gain by always increasing both its bids by  $\epsilon$ : The payments increases by at most  $2\epsilon$  but the winning probability increases by a constant. (AND now wins the atom.)

(2) Assume  $L_{and}^1 \geq 1$ . The AND must always bid 0 on item 2, since if it bids higher, when OR bids low on 2 (which happens with positive probability by Lemma 6) AND has negative utility. In this case OR gains by bidding  $\epsilon$  on item 2 and 0 on item 1. It will always win item 2 and pay  $\epsilon$  whereas previously OR paid at least 1 when it won item 1, which occurred with constant probability.  $\square$

We now show that the low values of AND are indeed zero.

**Lemma 8.**  $L_{and}^1 = L_{and}^2 = 0$ .

*Proof.* Assume by way of contradiction that  $L_{and}^1 > 0$ , and let us look at the relation between  $L_{and}^2$  and  $L_{or}^2$ . By the previous lemma if  $L_{and}^2 = L_{or}^2 = l$  then at most one of the players can have an atom at  $l$ , so we are left with two cases.

CASE I: If  $L_{and}^2 > L_{or}^2$  or  $L_{and}^2 = L_{or}^2$  but AND has no atom at  $L_{and}^2$ . Consider the expected utility of OR when it bids  $x \in [L_{or}^2, L_{or}^2 + \epsilon]$  on item 2 (with a low bid on the first item). OR probability of winning with such bids goes to zero as  $\epsilon$  goes to zero. Therefore OR's utility goes to zero, in contradiction to Lemma 3, which shows that OR has a strictly positive utility.

CASE II: If  $L_{and}^2 < L_{or}^2$  or  $L_{and}^2 = L_{or}^2$  but OR has no atom at  $L_{and}^2$ . Let us consider the set  $B(\epsilon)$  of all the bids of AND which are coordinate-wise no larger than  $(L_{and}^1 + \epsilon, L_{and}^2 + \epsilon)$  as  $\epsilon$  approaches zero. First note that since  $L_{and}^1 > 0$  and since by Lemma 6 OR bids low on item 1 with constant probability, the expected payment of AND on  $B(\epsilon)$  is some positive constant which is independent of  $\epsilon$ . On the other hand we show that the probability that AND wins with a bid in  $B(\epsilon)$  approaches 0 as  $\epsilon$  approaches zero. This implies that there exists an  $\epsilon > 0$  such that the AND has negative utility for  $B(\epsilon)$  and hence a contradiction.

Consider the probability of AND winning both items with a bid in  $B(\epsilon)$ . By Lemma 6 this is the sum of the probability that AND wins both items when OR bids low on item 1 and the probability that AND wins both items when OR bids low on item 2. We estimate these probabilities in the following cases.

(IIa) If OR bids low on the first item, then the probability AND wins the second item with a bid in  $B(\epsilon)$  goes to zero with  $\epsilon$  by our assumption that  $L_{and}^2 < L_{or}^2$  or  $L_{and}^2 = L_{or}^2$  but OR has no atom at  $L_{and}^2$ .

(IIb) If OR bids low on the second item, then it cannot bid low on the first item so it bids at least  $L_{and}^1$  on item 1. If AND does not have an atom at  $L_{and}^1$ , then OR bids strictly above  $L_{and}^1$  ( $L_{and}^1$  is a low value), therefore, AND's winning probability with  $B(\epsilon)$  goes to zero with  $\epsilon$ . If AND has an atom at  $L_{and}^1$ , by Lemma 7, OR cannot have an atom at  $L_{and}^1$  so the probability that OR bids less than  $L_{and}^1 + \epsilon$  goes to zero as  $\epsilon$  approaches zero, and hence the probability of AND winning the first item goes to zero. Hence, we have established that the probability that AND wins goes to zero as  $\epsilon$  approaches zero.  $\square$

By Lemma 8, any positive bid of OR has a positive probability of winning, hence the only low bids of OR is zero.

**Corollary 4.** *OR always bids either  $(x, 0)$  with  $x > 0$  or  $(0, y)$  with  $y > 0$ . Both of these events happen with positive probability.*

### 3.5 AND Bids on the Diagonal

In this section we are considering an equilibrium  $(\bar{F}_{and}, F_{or})$  of the AND-OR game, where  $\bar{F}_{and}$  maximally correlated.

**Lemma 9.**  $Pr_{(x,y) \sim \bar{F}_{and}}[x \neq y] = 0$ .

*Proof.* By way of contradiction, without loss of generality assume  $Pr_{(x,y) \sim \bar{F}_{and}}[x < y] > 0$ . So for some  $0 < b < c < H$  we will have  $Pr[\text{AND bids in } [0, b) \times (c, H)] > 0$  (since the former event is the union over the countable choices of the latter over all rationals  $0 < b < c < H$ ). Now since  $\bar{F}_{and}$  is maximally correlated, either

$\bar{F}_{and}(b, c) = \bar{F}_{and}(b, H)$  or  $\bar{F}_{and}(b, c) = \bar{F}_{and}(H, c)$ . However, the former possibility is in contradiction to  $Pr[\text{AND bids in } [0, b] \times (c, H)] > 0$  so we must have  $\bar{F}_{and}(b, c) = \bar{F}_{and}(H, c)$ . This means that whenever AND bids at most  $c$  on the second item, it also bids at most  $b$  on the first item.

In such a case, any bid  $(0, y)$  for OR, where  $b < y < c$  is strictly dominated the bid  $(b, 0)$  (since OR wins at least whenever  $(0, y)$  wins, pays strictly less, and this happens with positive probability since  $L_{and}^2 = 0$ ). Therefore the OR never bids  $(0, y)$  where  $b < y < c$ .

However AND does bid at least  $c$  on the second item with positive probability. This contradicts Lemma 2.  $\square$

### 3.6 The exact forms

In this section we show that the previous lemma imply that an equilibrium  $(\bar{F}_{and}, F_{or})$  of the AND-OR game with  $\bar{F}_{and}$  maximally correlated, must have a specific form.

**Lemma 10.**  $F_{or}(x, y) = \frac{1}{2}x/(1-x) + \frac{1}{2}y/(1-y)$  for  $0 \leq x, y \leq 1/2$ .

*Proof.* We know that OR bids 0 on exactly one of the items (Corollary 4) so  $F_{or}(x, y) = F_{or}(x, 0) + F_{or}(0, y)$ . Let  $\alpha = F_{or}(0, h)$  be the probability that OR bids 0 on the first item, and let  $1 - \alpha = F_{or}(h, 0)$ . Assume that AND bids  $(x, y)$ . With probability  $\alpha$  OR bids 0 on the first item and then 1) AND wins the first item and pays  $x$  for it, and 2) AND wins the second item with probability  $F_{or}(0, y)$ . Similarly, with probability  $1 - \alpha$  OR bids 0 on the second item and then AND 1) wins the second item and pays  $y$  for it, and 2) wins the first item with probability  $F_{or}(x, 0)$ . So we conclude that  $u_{AND}(x, y) = F_{or}(x, 0)(1 - x) + F_{or}(0, y)(1 - y) - \alpha x - (1 - \alpha)y$ .

Now notice that  $u_{AND}(x, y)$  is the sum of a function  $g_1(x) = F_{or}(x, 0)(1 - x) - \alpha x$ , that depends only on  $x$  and a function  $g_2(y) = F_{or}(0, y)(1 - y) - (1 - \alpha)y$  that depends only on  $y$ . We claim that  $g_1(x)$  must be a constant for all  $x \in [0, h]$  and  $g_2(y)$  must be a constant for all  $y \in [0, h]$ . We prove this claim for  $g_1$ , the proof for  $g_2$  is the same. Assume for a contradiction that  $g_1(x_1) > g_1(x_2)$  for some  $x_1, x_2 \in [0, h]$ . Then the bid  $(x_1, x_2)$  of AND strictly dominates the bid  $(x_2, x_2)$  in contradiction to the fact that  $(x_2, x_2)$  is a best response of AND for  $F_{or}$  which follows from Lemma 2 (the support of AND is an interval), Lemma 8 (the interval starts at 0), and Lemma 9 (AND bids on the diagonal).

As OR does not have an atom at  $(0, 0)$ ,  $F_{or}(0, y)$  and  $F_{or}(x, 0)$  approach 0 as  $x$  and  $y$  approach 0, respectively. Therefore  $u_{AND}(x, y)$  approaches 0 as  $x$  and  $y$  approach 0. It follows that we must have that  $g_1(x) = F_{or}(x, 0)(1 - x) - \alpha x = 0$  for all  $0 \leq x \leq h$ , and similarly  $g_2(y) = 0$  for all  $0 \leq y \leq h$ . I.e.  $F_{or}(x, 0) = \alpha x/(1 - x)$  and  $F_{or}(0, y) = (1 - \alpha)y/(1 - y)$  for all  $0 \leq x, y \leq h$ . In particular since  $F_{or}(h, h) = F_{or}(0, h) + F_{or}(h, 0) = 1$ , we have that  $\alpha h/(1 - h) + (1 - \alpha)h/(1 - h) = h/(1 - h) = 1$  which implies that  $h = 1/2$ . But then substituting  $h = 1/2$  into the expression we get  $F_{or}(0, h) = F_{or}(0, 1/2) = 1 - \alpha$ . But by its definition  $\alpha = F_{or}(0, h)$  so  $\alpha = 1/2$  and the lemma follows.  $\square$

**Lemma 11.**  $\bar{F}_{and}(x, y) = \frac{v-1/2}{v-\min(x, y)}$ .

*Proof.* Since AND bids on the diagonal, clearly  $\bar{F}_{and}(x, y) = \bar{F}_{and}(\min(x, y), \min(x, y))$ , so it suffices to characterize  $\bar{F}_{and}(x, x)$  for all  $x$ . The utility for OR for bidding  $(x, 0)$  is  $(v - x)\bar{F}_{and}(x, x)$ . By Corollary 4 (lowest bid of OR is 0), Corollary 2 (OR and AND have the same highest bid), Lemma 10 (this highest bid is  $1/2$ ), and Lemma 2 (OR bids in an interval) we know that the support of OR is the interval  $(0, 1/2)$ . So for every  $0 < x < 1/2$ ,  $(x, 0)$  is a best response to AND and thus  $(v - x)\bar{F}_{and}(x, x)$  is a constant independent of  $x$ . For  $x = 1/2$  we know that  $\bar{F}_{and}(1/2, 1/2) = 1$  and thus this constant is  $v - 1/2$ . It follows that  $(v - x)\bar{F}_{and}(x, x) = v - 1/2$  for all  $x$ , i.e.,  $\bar{F}_{and}(x, x) = (v - 1/2)/(v - x)$ , and the lemma follows.  $\square$

### 3.7 Completing the Proof

Now let us put everything together to prove Theorem 1.

*Proof of Theorem 1:* We start by showing the necessary conditions for an equilibrium. Take an equilibrium  $(F_{and}, F_{or})$ . As Lemma 5 shows  $(\bar{F}_{and}, F_{or})$  is also an equilibrium, with the same allocations, payments, and utilities. From Lemmas 10 and 11 we have that  $F_{or}(x, y) = (x/(1 - x) + y/(1 - y))/2$  and  $\bar{F}_{and}(x, y) = \min(F_{and}(x, H), F_{and}(H, y)) = (v - 1/2)/(v - \min(x, y))$ . By Lemma 4,  $F_{and}$  is dominated by  $\bar{F}_{and}$ .

To show that  $F_{and}(0, 0) = \bar{F}_{and}(0, 0)$ , assume for a contradiction that  $F_{and}(0, 0) \neq \bar{F}_{and}(0, 0)$ . Since by definition  $\bar{F}_{and}(x, y) \geq F_{and}(x, y)$ , we have that  $F_{and}(0, 0) < \bar{F}_{and}(0, 0) = (v - 1/2)/v$ . For  $x, y > 0$  the utility of OR from the bid  $(x, y)$  is  $u_{or}(x, y) = F_{and}(x, H)(v - x) + F_{and}(H, y)(v - y) - F_{and}(x, y) \cdot v$  (for  $x = 0$  or  $y = 0$  the atom of  $F_{and}$  at  $x = 0$  and  $y = 0$  may cause the utility to be lower depending on the tie breaking rule). Now let  $x$  and  $y$  approach 0 and we get at the limit a utility of  $v \cdot (F_{and}(0, H) + F_{and}(H, 0) - F_{and}(0, 0)) > v \cdot (\bar{F}_{and}(0, H) + \bar{F}_{and}(H, 0) - \bar{F}_{and}(0, 0)) = v \cdot \bar{F}_{and}(0, 0) = v - 1/2$  (where the inequality follows since the marginal distributions of  $F_{and}$  and  $\bar{F}_{and}$  are the same). Thus for small enough  $x$  and  $y$  the utility of OR is strictly greater than  $v - 1/2$ . This however contradicts our derivation of  $F_{or}$  whose support includes the bid  $(0, 1/2)$  for which  $u_{or}(0, 1/2) = v - 1/2$ , a contradiction to it being a best response to  $F_{and}$ . Therefore,  $F_{and}(0, 0) = \bar{F}_{and}(0, 0)$ .

We now show the sufficient conditions for an equilibrium. For the fixed distributions  $\bar{F}_{and}(x, y) = (v - 1/2)/(v - \min(x, y))$  and  $F_{or}(x, y) = (x/(1 - x) + y/(1 - y))/2$  one may directly verify that  $(\bar{F}_{and}, F_{or})$  is an equilibrium (as was shown in [3]). Now take  $F_{and}$  such that  $\bar{F}_{and}(x, y) = (v - 1/2)/(v - \min(x, y))$ ,  $F_{and}(0, 0) = (v - 1/2)/v$ , and  $F_{and}(x, 0) = F_{and}(0, 0) = F_{and}(0, y)$ .<sup>5</sup> We first need to show that  $F_{and}$  is also a best response to  $F_{or}$  (and not just  $\bar{F}_{and}$ ), that is we need to show that  $u_{and}(F_{and}, F_{or}) = u_{and}(\bar{F}_{and}, F_{or})$ . By Proposition 2 this will happen whenever the probability that AND wins no item is the same in

<sup>5</sup> This follows since  $F_{and}^*(0, 0) = F_{and}(0, 0) \leq F_{and}(0, y) \leq F_{and}(0, H) = F_{and}^*(0, H) = F_{and}^*(0, 0)$ .

both cases. Since  $F_{and}(x, 0) = F_{and}(0, 0) = F_{and}(0, y)$ , the probability for AND winning no item is exactly the probability that it bids  $(0, 0)$  times the probability that OR wins the tie at its 0 bid, which is the same in  $F_{and}$  and  $\bar{F}_{and}$  since  $F_{and}(0, 0) = \bar{F}_{and}(0, 0)$ .

Finally we need to show that  $F_{or}$  is also a best response to  $F_{and}$ . As before, for  $0 < x, y$ , the utility of OR from a bid  $(x, y)$  is  $u_{or}(x, y) = F_{and}(x, H)(v - x) + F_{and}(H, y)(v - y) - F_{and}(x, y)v$ . Notice that  $F_{and}(x, H) = \bar{F}_{and}(x, H) = (v - 1/2)/(v - x)$  so the first and the second terms equal the constant  $v - 1/2$ . It follows that the maximum utility is obtained as  $x$  and  $y$  approach 0 (since this minimizes the last term  $F(x, y)$ ). The utility of OR when  $x$  and  $y$  approach 0, approaches (from below)  $2v - 1 - F_{and}(0, 0)v = 2v - 1 - \bar{F}_{and}(0, 0)v = (2v - 1) - (v - 1/2) = v - 1/2$ . Since  $F_{and}(x, 0) = F_{and}(0, 0) = F_{and}(0, y)$ , we have that the utility of OR is  $v - 1/2$  at any point in the support of  $F_{or}^*$ , and hence it is a best response.  $\square$

## 4 Properties of the equilibrium

We present few properties of the Nash equilibrium in Theorem 1. Our analysis is a function of the value  $v$  (of the OR player). We analyze the probability that each player wins, the expected revenue and the expected social welfare. By Theorem 1 all these quantities are identical in every Nash equilibrium. (The proofs and the figures are in the Appendix.)

First, we derive the probability that the AND player wins (clearly the probability that the OR player wins is the complement). This probability is depicted in Figure 1.

**Lemma 12.** *The probability that the AND player wins is  $\frac{\ln(2) - \frac{1}{2}}{v} + O(\frac{1}{v^2})$  and for  $v = 1$  this probability is  $1/4$ .*

Next we compute the expected revenue. The expected utility of the AND player is 0 and therefore the revenue from the AND player equals to the probability that it wins. It remains to compute the revenue from the OR player.

**Theorem 2.** *The expected revenue from the OR player is  $1 - \ln 2 - O(\frac{1}{v})$ . For  $v = 1$  the expected revenue from the OR player is  $1/4$ .*

The revenue from the OR player is plotted in Figure 2 as a function of the value  $v$ . The revenue from the auction (i.e., sum of both players) is shown in Figure 3.

Using the probability that each player wins, we can compute the expected social welfare, which is  $(\Pr[\text{And wins}] + v \cdot \Pr[\text{OR wins}])$ .

**Theorem 3.** *The expected social welfare is  $v - \ln(2) + 1/2 + \frac{\ln(2) - 1/2}{v} + O(1/v^2)$ .*

Figure 4 shows the Price of Anarchy of the equilibrium. That is we divide the expected social welfare in equilibrium  $(\Pr[\text{And wins}] + v \cdot \Pr[\text{OR wins}])$  by the maximum social welfare, that is  $\max\{v, 1\}$ . The difference  $\max\{v, 1\} - (\Pr[\text{And wins}] + v \cdot \Pr[\text{OR wins}])$  is shown in Figure 5. The expected loss converges to  $\ln(2) - 0.5 \approx 0.19$  as the value  $v$  of the OR player goes to infinity.

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## B Missing Proofs

*Proof of Lemma 12:* If  $v \neq 1$  we get that

$$\Pr[\text{AND wins}] = \int_0^{1/2} F'_{and}(x) F_{or}(x) dx$$

$$\begin{aligned}
&= \int_0^{1/2} \frac{v - \frac{1}{2}}{(v-x)^2} \frac{x}{1-x} dx \\
&= \left(v - \frac{1}{2}\right) \left[ \frac{\ln \frac{v-x}{1-x}}{(v-1)^2} - \frac{v}{(v-1)(v-x)} \right]_0^{1/2} \\
&= \left(v - \frac{1}{2}\right) \left[ \frac{\ln(2v-1)}{(v-1)^2} - \frac{v}{(v-1)(v-1/2)} - \frac{\ln v}{(v-1)^2} + \frac{1}{v-1} \right] \\
&= \left(v - \frac{1}{2}\right) \left[ \frac{\ln(2v-1) - \ln v}{(v-1)^2} - \frac{1/2}{(v-1)(v-1/2)} \right] \\
&= \frac{(v-1/2) \ln(2 - \frac{1}{v}) - \frac{1}{2}(v-1)}{(v-1)^2} \\
&= \frac{\ln(2) - \frac{1}{2}}{v} + O\left(\frac{1}{v^2}\right)
\end{aligned}$$

For  $v = 1$  a similar calculation shows that

$$\Pr[\text{AND wins} \mid v = 1] = \frac{1}{2} \int_0^{1/2} \frac{x}{(1-x)^3} dx = \frac{1}{2} \left[ \frac{2x-1}{2(x-1)^2} \right]_0^{1/2} = \frac{1}{4}.$$

□

*Proof of Theorem 2:*

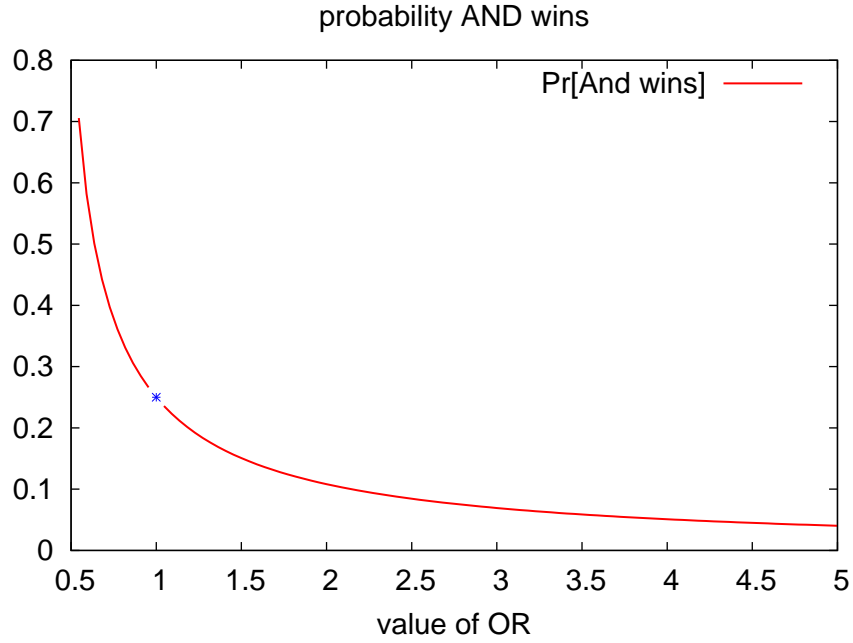
$$\begin{aligned}
\text{Revenue}(\text{OR}) &= \int_0^{1/2} x F'_{or}(x) F_{and}(x) dx \\
&= \int_0^{1/2} x \frac{1}{(1-x)^2} \frac{(v-\frac{1}{2})}{(v-x)} dx \\
&= \frac{(v-\frac{1}{2})}{(v-1)^2} \left[ v \ln \left( \frac{1-x}{v-x} \right) + \frac{(v-1)}{(1-x)} \right]_0^{1/2} \\
&= \frac{(v-\frac{1}{2})}{(v-1)^2} \left[ v \ln \left( \frac{\frac{1}{2}}{v-\frac{1}{2}} \right) + \frac{(v-1)}{1/2} - \left( v \ln \frac{1}{v} + (v-1) \right) \right] \\
&= \frac{(v-\frac{1}{2})}{(v-1)^2} \left[ v-1 - v \ln 2 + v \ln \frac{v}{v-\frac{1}{2}} \right] \\
&= \frac{(v-\frac{1}{2})}{(v-1)^2} \left[ v-1 - v \ln 2 + v \ln \left( 1 + \frac{\frac{1}{2}}{v-\frac{1}{2}} \right) \right] \\
&= \frac{(v-\frac{1}{2})}{(v-1)^2} \left[ v-1 - (v-1) \ln 2 - \ln 2 + v \ln \left( 1 + \frac{1}{2v-1} \right) \right] \\
&= (1 - \ln 2) \frac{v-\frac{1}{2}}{v-1} - O\left(\frac{1}{v}\right) \\
&= 1 - \ln 2 - O\left(\frac{1}{v}\right) = 0.3068 - O\left(\frac{1}{v}\right)
\end{aligned}$$

For  $v = 1$  a similar calculation shows that the revenue of the OR player is 0.25. □

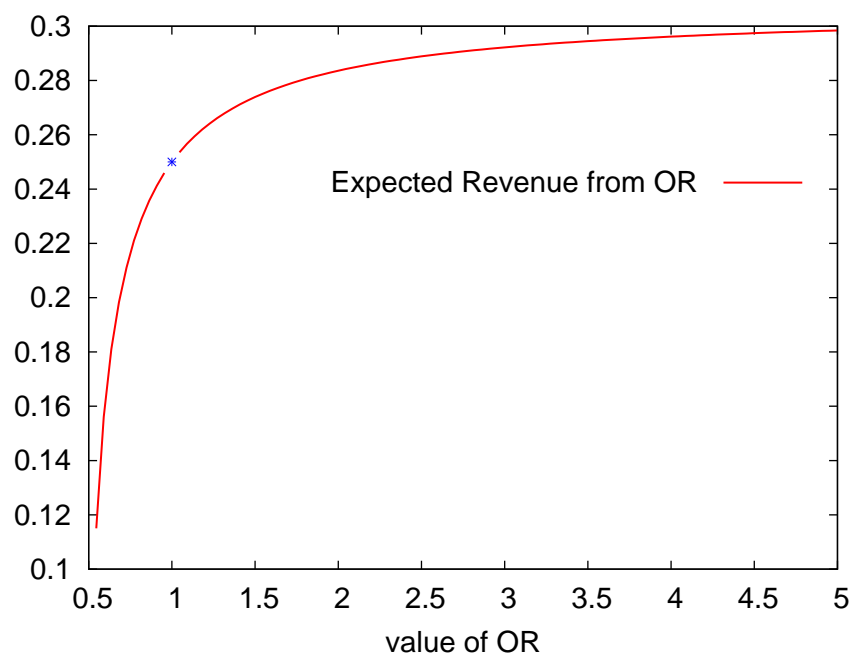


## C Missing Figures

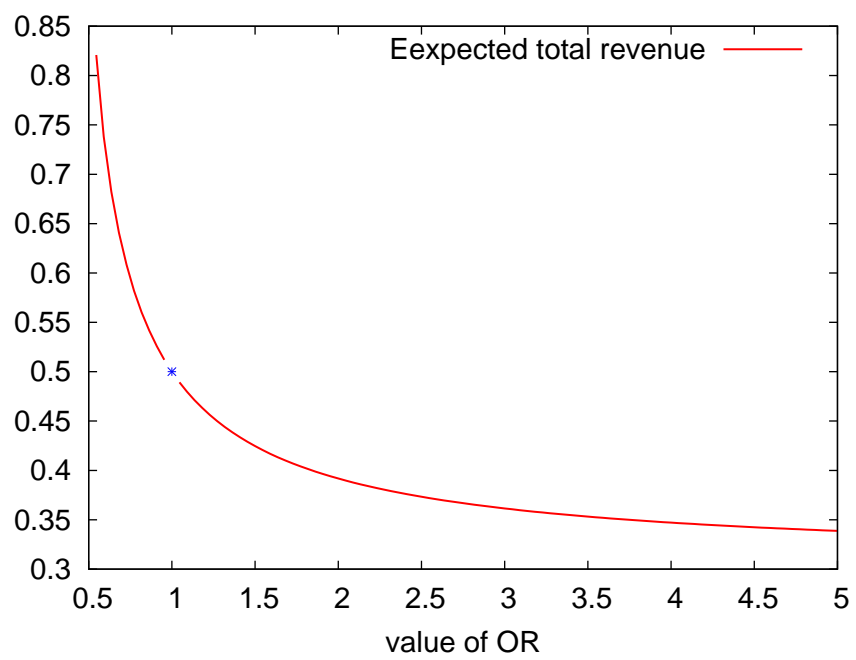
In this section we present the missing figures. All the figures depict properties of the equilibrium, as a function of the value of the OR player; the value of the AND player is taken to be 1. The graphs are only presented for  $v > 0.5$ , as otherwise there is a Walrasian equilibrium. In all the figures the asterisk depicts the crossover point, in which the value of the OR player is 1.



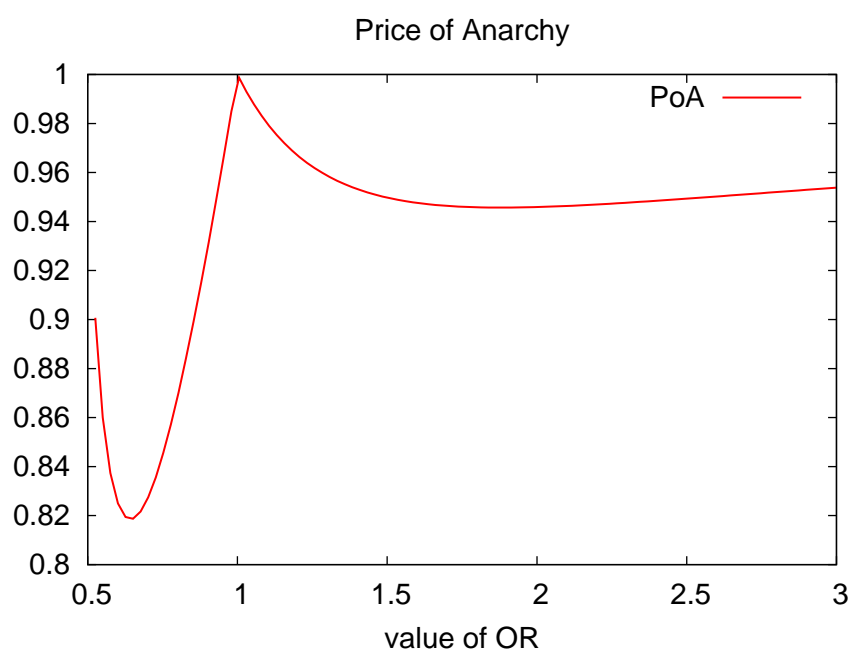
**Fig. 1.** The probability that the AND player wins which is the same as the revenue generated from the AND player.



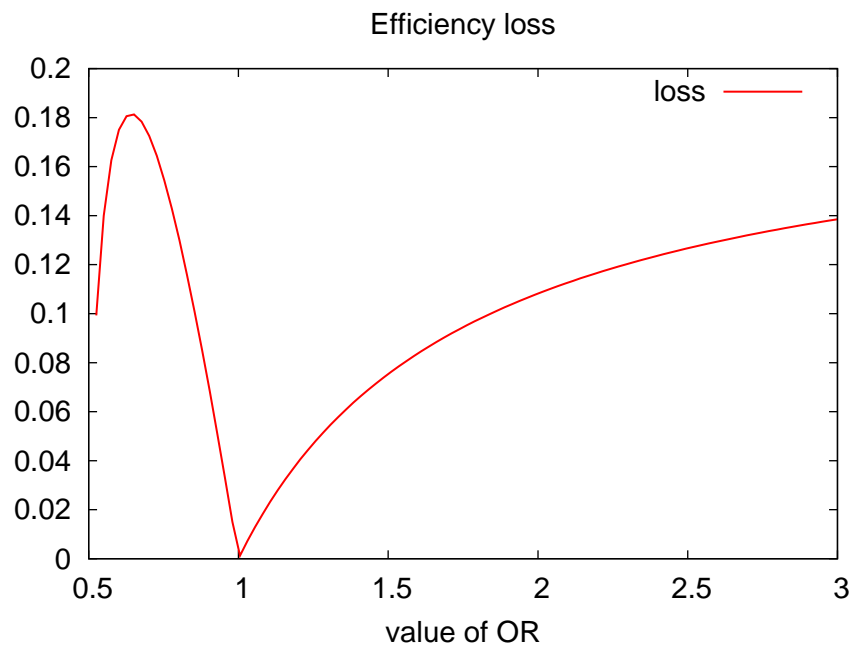
**Fig. 2.** The revenue from the OR player.



**Fig. 3.** The revenue from the OR player and the AND player.



**Fig. 4.** The Price of Anarchy of the AND/OR game. This is the fraction of the optimal social welfare obtained by the auction. For  $v < 1$  it achieves a minimum of  $\approx 0.818485$  at  $v \approx 0.643028$ . For  $v > 1$  it achieves a minimum of  $\approx 0.945682$  at  $v \approx 1.87999$ .



**Fig. 5.** The additive loss in social welfare of the Nash equilibrium of the **AND/OR** game. That is, the optimal social welfare minus the social welfare of the Nash equilibrium.